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100 Excellent

Exam I, MTH 205, Fall 2014

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$(-\infty, 12)$

$12 - x > 0 \quad x \neq 4$
$12 > x$
$x < 12$

QUESTION 1. (6 points) Find the largest interval around x so that the LDE: $\frac{\sqrt{x-4}}{\sqrt{12-x}}y^{(3)} + \frac{x-1}{x-7}y' + 3y = x^2 + 13$, $y^{(2)}(5) = y'(5) = 7$, and $y(5) = -6$ has a unique solution.

$\frac{\sqrt{x-4}}{\sqrt{12-x}}$ $\quad 12 - x > 0 \quad (-\infty, 12)$

$12 > x \quad (-\infty, 4) \cup (4, 12)$

$x < 12$

$x - 7$

$(-\infty, 7) \cup (7, \infty)$

$\mathbb{R}$

$\boxed{I = (4, 7)}$

$\int_0^x 2 \sin u \, du.$

$(-2\cos u)_0^x$

$-2\cos x + 2$

QUESTION 2. (10 points) Solve for $x(t), y(t)$

$x'(t) - y(t) = 2$

$x(t) + y'(t) = 2$, where $x(0) = 2, y(0) = -1, x'(0) = 1, y'(0) = 0$

$sX(s) - x(0) - Y(s) = \frac{2}{s}$

$sX(s) - 2 - Y(s) = \frac{2}{s}$

$\boxed{sX(s) - Y(s) = \frac{2 + 2s}{s}}$

$X(s) + sY(s) + 1 = \frac{2}{s}$

$\boxed{X(s) + sY(s) = \frac{2 - s}{s}}$

$sX(s) - \frac{2 + 2s}{s} = Y(s)$

$X(s) + s^2 X(s) - 2 - 2s = \frac{2 - s}{s}$

$X(s)(1 + s^2) = \frac{2 - s}{s} + 2 + 2s$

$= \frac{2 - s + 2s + 2s^2}{s}$

$X(s) = \frac{2s^2 + s + 2}{s(1 + s^2)}$

$= \frac{2s}{(s^2+1)} + \frac{1}{(s^2+1)} + \frac{2}{s(s^2+1)}$

$x(t) = 2\cos t + \sin t + 2 * \sin t$

$= 2\cos t + \sin t - 2\cos t + 2 = \boxed{\sin t + 2}$

$x'(t) = \cos t$

$\boxed{\cos t - 2 = y(t)}$

QUESTION 3. (30 points, each is 6 points)

(i) Find $\ell^{-1}\{\frac{s^3+24}{s^5}\}$ $\ell^{-1}\left\{\frac{1}{s^2}+\frac{24}{s^5}\right\}$

$= x + \frac{24}{4!}x^4 = x + x^4$

(ii) Find $\ell^{-1}\{\frac{e^{-2s}}{(s+4)^2+4}\}$ $= \ell^{-1}\left\{e^{-2s}\left(\frac{1}{(s+4)^2+4}\right)\right\} = \frac{1}{2}u(x-2)\sin 2(x-2)e^{-4(x-2)}$

$f(x+2) = \frac{1}{2}\ell^{-1}\left\{\frac{2}{(s+4)^2+4}\right\} = \frac{1}{2}\sin 2x\ e^{-4x}$

$f(x) = \frac{1}{2}\sin 2(x-2)\,e^{-4(x-2)}$

(iii) Find $\ell\{u(x-1)e^{(x-1)}sin(x-1)\}$ $= e^{-s}\,\ell\{e^x \sin x\} =$

$= e^{-s}\frac{1}{(s-1)^2+1}$

(iv) Find $\ell^{-1}\{\frac{s+2}{s^2+4s+5}\}$ $\ell^{-1}\left\{\frac{s+2}{(s^2+4s+4)-4+5}\right\}$ $\ell^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$

$= e^{-2x}\cos x$

$x-r$

$e^{2x-2r}e^{r}$

(v) Find $\ell\{\int_0^x e^{2x-r}sin(r)\,dr\}$ $= \ell\left\{\int_0^x e^{2(x-r)}(\sin r e^r)\,dr\right\}$

$= \ell\left\{(e^{2x} * (\sin x)e^x)\right\}$

$= \left(\frac{1}{s-2}\right)\left(\frac{1}{(s-1)^2+1}\right)$

QUESTION 4. (54 points, each is 9 points) Use any method you want to solve for $y(x)$:

(i) $y^{(2)} - 2y' + y = u(x-1)e^{(x-1)}$ [Here you need to find y_g]. $y_h = c_1 e^x + c_2 x e^x$

y_h $m^2 - 2m + 1 = 0$ $m = 1$, $m = 1$

y_p $Y(s)\left[(s-1)^2\right] = e^{-s} \mathcal{L}\{e^x\} = \frac{e^{-s}}{(s-1)}$

$Y(s) = \frac{e^{-s}}{(s-1)^3}$ | $y_p = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{(s-1)^3}\right)\right\}$

$\boxed{= \frac{1}{2}u(x-1)(x-1)^2 e^{x-1}}$

$f(x+1) = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{(s-1)^3}\right\} = \frac{1}{2}x^2 e^{+x}$

$f(x) = \frac{1}{2}(x-1)^2 e^{x-1}$

$y_g = c_1 e^x + c_2 x e^x + \frac{1}{2}u(x-1)(x-1)^2 e^{x-1}$

(ii) $y^{(6)} + 5y^{(5)} + 4y^{(4)} = 30e^{-4x}$ [here you need to find y_g].

y_h

$m^6 + 5m^5 + 4m^4 = 0$

$m^4(m^2 + 5m + 4) = 0$

$m=0$, $m=0$, $m=0$, $m=0$, $m=-1$, $m=-4$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-x} + c_6 e^{-4x}$

$Y(s)\left[s^4(s+1)(s+4)\right] = \frac{30}{s+4}$

$Y(s) = \frac{30}{(s+4)^2(s+1)s^4}$

$= \cancel{\frac{a}{(s+4)}} + \frac{b}{(s+4)^2} + \cancel{\frac{c}{(s+1)}} + \cancel{\frac{d}{s}} + \cancel{\frac{e}{s^2}} + \cancel{\frac{g}{s^3}} + \cancel{\frac{h}{s^4}}$

$b = -\frac{5}{128}$

$y_p = \mathcal{L}^{-1}\left\{-\frac{5}{128}\frac{1}{(s+4)^2}\right\}$

$= -\frac{5}{128}xe^{-4x}$

$y_g = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-x} + c_6 e^{-4x} - \frac{5}{128}xe^{-4x}$

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(iii) $y^{(2)} + \int_0^x y(r)e^{x-r}\,dr = \int_0^x (x-r)e^{r}\,dr, y(0) = 0, y'(0) = 1$

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}^{1} + Y(s)\left(\frac{1}{s-1}\right) = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

$$Y(s)\left(\frac{1}{s-1} + s^2\right) - 1 = \frac{1}{s^2(s-1)}$$

$$Y(s)\left(\frac{1+s^2(s-1)}{\cancel{(s-1)}}\right) = \frac{1+s^2(s-1)}{s^2\cancel{(s-1)}}$$

$$Y(s) = \frac{\cancel{(1+s^2(s-1))}}{s^2\cancel{(1+s^2(s-1))}} = \frac{1}{s^2}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x$$

(iv) $y^{(2)} + 2y' + 2y = xe^{-x}, y(0) = 0$ and $y'(0) = 1$. [Hint: note that by completing the square method we have $s^2 + bs + c = (s+b/2)^2 + c - b^2/4$ and $\frac{e}{f} + d = \frac{e+fd}{f}$]

$$s^2Y(s) - \cancel{sy(0)} - y'(0) + 2sY(s) - 2\cancel{y(0)} + 2Y(s) = \frac{1}{(s+1)^2}$$

$$Y(s)[s^2+2s+2] - 1 = \frac{1+(s+1)^2}{(s+1)^2}$$

$$Y(s)[s^2+2s+1-1+2] = \frac{1+(s+1)^2}{(s+1)^2}$$

$$Y(s)\,\cancel{[(s+1)^2+1]} = \frac{\cancel{1+(s+1)^2}}{(s+1)^2}$$

$$Y(s) = \frac{1}{(s+1)^2}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = xe^{-x}$$

(v) $y^{(3)} - 4y^{(2)} + 5y' = 10$ [here you need to find y_g]

y_h $\quad m^3 - 4m^2 + 5m = 0$

$m(m^2 - 4m + 5) = 0$

$m = 0$

$m = 2 + i$

$m = 2 - i$

$y_h = c_1 + c_2 e^{2x} \cos x + c_3 e^{2x} \sin x$

y_p $\quad Y(s)[s(s^2 - 4s + 5)] = \frac{10}{s}$

$Y(s) = \cancel{\frac{a}{s}} + \frac{b}{s^2} + \cancel{\frac{c}{(s^2-4s+5)}}$

$b = 2$

$y(x) = 2x$

$y_g = c_1 + c_2 e^{2x} \cos x + c_3 e^{2x} \sin x + 2x$

(vi) Let $k(x) = 4xe^{3x}$. Consider the LDE: $y^{(2)} + ay' + by = k(x)$. Find a, b so that $y(x) = k(x) = 4xe^{3x}$ is the unique solution to the given LDE. [Hint: If you want to use Laplace, then since y(x) is given, you should be able to find $y(0)$ and $y'(0)$, anyway it is clear that $y(0) = 0, y'(0) = 4$].

$s^2 Y(s) - \cancel{s y(0)} - y'(0) + as Y(s) - \cancel{a y(0)} + b Y(s) = \frac{4}{(s-3)^2}$

$Y(s)[s^2 + as + b] = \frac{4}{(s-3)^2} + \frac{4(s-3)^2}{(s-3)^2} = \frac{4((s-3)^2 + 1)}{(s-3)^2 (s^2 + as + b)}$

$\frac{(s-3)^2 + 1}{(s^2 + as + b)} = 1$

$s^2 - 6s + 10 = s^2 + as + b$

$a = -6$

$b = 10$

Faculty information

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